Stochastic motion of charged particles in a magnetic field

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The collisional diffusion process is discussed in a model in which the motion of charged particles in a magnetic field is treated as a stochastic process similar to that of Brownian particles. Collisional diffusion coefficients are obtaiend, which are similar to those calculated through classical collisional theory.

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Up to now, explaining anomalous transport in magnetic confinement devices has been a problem. However, we would rather review the collisional transport processes from a different viewpoint, before discarding their contributions to anomalous transport.

Physical processes associated with collision may be too complicated to describe accurately. However, for the case of collision diffusion when external forces are absent except for a static magnetic field, we need not go into such detail to determine the velocity distribution functions of electrons, so the treatment will be easier than that by solving a Fokker-Planck equation [1].

For simplicity, consider a plasma composed of electrons and one single ion. We only discuss the case in which electrons collide with ions and vice versa. Collisions between electrons or ions are nelegected completely; this is regularly regarded as unimportant to collision diffusion.

In a dense plasma, an electron will undergo a large number of collisions (short-range Coulomb interactions) per unit time with ions in the Debye sphere, each lasting an extremely short time. Obviously, the total Coulomb force acting on this electron, denoted as $\mathbf{F}_C(t)$, is random, so we treat the motion of electrons as stochastic processes. When the plasma is in equilibrium, we will regard the motions of electrons to be the same, as a statistical average, because the velocity of an electron changes very rapidly.

Now, let us assume that $\mathbf{F}_C(t)$ is only correlated with itself over a intercollisional time τ_c, which is much shorter than the average collisional period. The time correlation function of $\mathbf{F}_C(t)$ is

$$\overline{F_i(t)F_j(t')}=A_{ij}\delta(t-t') \ . \tag{1}$$

Here, $F_{i\,(j)}$ represents the component of $\mathbf{F}_C$. Furthermore, we assume A_{ij} is independent in time, obviously, $A_{ij}=A_{ji}$.

In the direction of motion, electrons will collide with more ions, and we make an assumption that binary collision is dominant. This gives a frictional force

$$\mathbf{F}_f(t)=-\int M_{ei}\Delta\mathbf{V}_{ei}n_i v_{ei}\sigma_{ei}d\Omega \ . \tag{2}$$

Here M_{ei} is the reduced mass of the electron and ion, $\mathbf{V}_{ei}$ is their relative velocity, n_i is the density of the ions, and σ_{ei} is the Rutherford scattering cross section. Approximately, $\mathbf{V}_{ie}=\mathbf{V}_e$; integrating over the solid angle, we obtain from Eq. (2):

$$\mathbf{F}_f(t)=\gamma_{ei}m_e\mathbf{V}_e \ . \tag{3}$$

Here γ_{ei} is the familiar collisional frequency,

$$\gamma_{ei}=\frac{4\pi Z^2e^4n_i\ln\Lambda}{m_e^2V_t^3} \ ,$$

where V_t is the thermal speed of electrons and $\ln\Lambda$ is the Coulomb logarithm, $\ln\Lambda=\ln(\lambda_D/b_0)$, where λ_D is the Debye length and b_0 is the distance of closest approach of two particles. If the effect of magnetic field is taken into account, λ_d will be replaced by the gyroradius of the electron in the Coulomb logarithm, when the latter is significantly smaller than the former [2].

Thus the motion equation of the electrons is written as

$$m_e\frac{d\mathbf{V}_e}{dt}=-\gamma_{ei}m_e\mathbf{V}_e-\frac{e\mathbf{V}_e\times\mathbf{B}}{c}+\mathbf{F}_C \ . \tag{4}$$

This is the Langevin equation of electrons in the magnetic field. It is similar to that of Brownian particles.

In cylindrical coordinate, a static and homogeneous magnetic field is assumed, $\mathbf{B}=B_z\mathbf{e}_z+B_\theta\mathbf{e}_\theta$, and the statistical momentum tensor $\overline{\mathbf{A}}$ is written as

$$\overline{\mathbf{A}}=\begin{vmatrix} A_{zz} & A_{zr} & A_{z\theta} \\ A_{zr} & A_{rr} & A_{r\theta} \\ A_{z\theta} & A_{r\theta} & A_{\theta\theta} \end{vmatrix} \ .$$

Thus the solutions of Eq. (4) are

$$V_z=[C_1(t)\Omega_z+C_2(t)\Omega_\theta\cos(\Omega t)+C_3(t)\Omega_\theta\sin(\Omega t)]e^{-\gamma_{ei}t} \ , \tag{5}$$

$$V_r=[-C_2(t)\Omega\sin(\Omega t)+C_3(t)\Omega\cos(\Omega t)]e^{-\gamma_{ei}t} \ , \tag{6}$$

$$V_\theta=[C_1(t)\Omega_\theta-C_2(t)\Omega_z\cos(\Omega t)-C_3(t)\Omega_z\sin(\Omega t)]e^{-\gamma_{ei}t} \ , \tag{7}$$

where $C_1(t)$, $C_2(t)$, and $C_3(t)$ are, respectively,

$$C_1(t)=\frac{1}{\Omega^2 m_e}\int_{-\infty}^{t}(F_z\Omega_z+F_\theta\Omega_\theta)e^{\gamma_{ei}\xi}d\xi\ , \tag{8}$$

$$C_2(t)=\frac{1}{\Omega^2 m_e}\int_{-\infty}^{t}[(F_z\Omega_\theta-F_\theta\Omega_z)\cos(\Omega\xi)-F_r\Omega\sin(\Omega\xi)]e^{\gamma_{ei}\xi}d\xi\ , \tag{9}$$

$$C_3(t)=\frac{1}{\Omega^2 m_e}\int_{-\infty}^{t}[(F_z\Omega_\theta-F_\theta\Omega_z)\sin(\Omega\xi)+F_r\Omega\cos(\Omega\xi)]e^{\gamma_{ei}\xi}d\xi\ . \tag{10}$$

Here (F_z,F_r,F_θ) is the component of $\mathbf{F}_C$, $\Omega_\theta=-eB_\theta/(m_e c),\Omega_z=-eB_z/(m_e c)$, $\Omega=(\Omega_\theta^2+\Omega_z^2)^{0.5}$.

Generally, $\Omega\gg\gamma_{ei}$. From Eqs. (1) and (5)–(10), we get the correlation functions of $\mathbf{V}_e(t)$ to zero order:

$$\overline{V_z(t)V_z(t')}=\frac{\Omega_z^2 e^{-\gamma_{ei}|t-t'|}}{2\gamma_{ei}\Omega^4 m_e^2}(A_{zz}\Omega_z^2+A_{\theta\theta}\Omega_\theta^2+2A_{z\theta}\Omega_\theta\Omega_z)\ , \tag{11}$$

$$\overline{V_r(t)V_r(t')}=\frac{e^{-\gamma_{ei}|t-t'|}}{4\gamma_{ei}\Omega^2 m_e^2}(A_{zz}\Omega_\theta^2+A_{\theta\theta}\Omega_z^2+A_{rr}\Omega^2-2A_{z\theta}\Omega_\theta\Omega_z)\cos\Omega(t-t')\ , \tag{12}$$

$$\overline{V_\theta(t)V_\theta(t')}=\frac{\Omega_z^2 e^{-\gamma_{ei}|t-t'|}}{2\gamma_{ei}\Omega^u m_e^2}(A_{zz}\Omega_z^2+A_{\theta\theta}\Omega_\theta^2+2A_{z\theta}\Omega_\theta\Omega_z)+\frac{e^{-\gamma_{ei}|t-t'|}}{4\gamma_{ei}\Omega^2 m_e^2}(A_{zz}\Omega_\theta^2+A_{\theta\theta}\Omega_z^2+A_{rr}\Omega^2-2A_{z\theta}\Omega_\theta\Omega_z)\cos\Omega(t-t')\ . \tag{13}$$

The velocity distribution function of electrons is assumed to be Maxwellian, so we have

$$V_z^2=V_r^2=V_\theta^2=\tfrac{1}{2}V_t^2\ . \tag{14}$$

The diffusion coefficient is given by [3]

$$D=\lim_{T\to\infty}\frac{1}{T}\int_0^T d\xi'\int_0^T\overline{V(\xi)V(\xi')}d\xi\ . \tag{15}$$

We use Eqs. (11)–(15) to obtain diffusion coefficients in each direction:

$$D_z=\frac{\Omega_z^2 V_t^2}{\Omega^2\gamma_{ei}}\ , \tag{16}$$

$$D_r=\frac{\gamma_{ei}V_t^2}{\gamma_{ei}^2+\Omega^2}\ , \tag{17}$$

$$D_\theta=\frac{\Omega_\theta^2 V_t^2}{\Omega^2\gamma_{ei}}\ . \tag{18}$$

The results are similar to those calculated by classical collisional theory [1].

In Brownian motion theory, diffusion of Brownian particles is only dependent on their dissipative processes in the liquid; so is the diffusion of charged particles in a magnetic field, as indicated by Eqs. (17)–(19), though diffusion will be restrained by the magnetic field. To see this, let us assume that there exists another dissipative channel of electrons (for instance, enhanced radiation, etc.) locally apart from collisional dissipation in the plasma. The region is denoted by D. The electrons in this region will endure an additional frictional force, which is simply written as $\mathbf{F}_L=-\gamma_L\mathbf{V}_e$; here γ_L represents energy dissipative rate of electrons. Adding this term in Eq. (4), and assuming γ_L to be constant, it is easy to obtain the diffusion coefficients of electrons in region D:

$$D_z=\frac{\Omega_z^2 V_t^2}{\Omega^2(\gamma_{ei}+\gamma_L)}\ , \tag{19}$$

$$D_r=\frac{(\gamma_{ei}+\gamma_L)V_t^2}{(\gamma_{ei}+\gamma_L)^2+\Omega^2}\ , \tag{20}$$

$$D_\theta=\frac{\Omega_\theta^2 V_t^2}{\Omega^2(\gamma_{ei}+\gamma_L)}\ . \tag{21}$$

The dissipative processes will enhance diffusion, though the treatment above is very crude.

The diffusion coefficients of ions are obtained by a similar procedure:

$$D_{iz}=\frac{\Omega_{iz}^2 V_i^2}{\Omega_i^2\gamma_{ie}}\ , \tag{22}$$

$$D_{ir}=\frac{\gamma_{ie}+V_i^2}{\gamma_{ie}^2+\Omega_i^2}\ , \tag{23}$$

$$D_{i\theta}=\frac{\Omega_{i\theta}^2 V_i^2}{\Omega_i^2\gamma_{ie}}\ . \tag{24}$$

Here

$$\Omega_{i\theta}=\frac{qB_\theta}{m_i c}\ ,$$

$$\Omega_{iz}=\frac{qB_z}{m_i c}\ ,$$

$$\Omega_i=(\Omega_{i\theta}^2+\omega_{iz}^2)^{0.5}\ ,$$

V_i is the ion's thermal speed, and γ_{ie} is the ion-electron collisional frequency.

It is worthwhile to note that the effect of other particles on a binary collision has been treated as a white noise, and the differential scattering cross section has not been effected by the external field, which would not be true for the existence of other dissipative mechanisms, or fluctuations when the density of plasma is very high. A more detailed consideration of this subject will be extremely valuable.

In conclusion, it has been shown that the motion of charged particles in a magnetic field can be regarded as a stochastic process. With this approach, it would be convenient to discuss other behaviors of charged particles in a magnetic field, particularly for the transport processes when there exist other stochastic fields, for instance, a stochastic magnetic field as discussed previously [4,5], which we will present later.

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